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Diophantine Equations Involving Sums of Fibonacci and Jacobsthal Numbers

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Abstract

This study identifies all Fibonacci numbers that can be expressed as the sum of two Jacobsthal numbers and all Jacobsthal numbers that can be expressed as the sum of two Fibonacci numbers. More precisely, we identify every non-negative integer solution $(a,\ b,\ c)$ of the Diophantine equations $F_a + F_b = J_c$ and $J_a + J_b = F_c$, where $\{F_c\}_{c\geq 0}$ and $\{J_c\}_{c\geq 0}$ are the sequences of Fibonacci and Jacobsthal numbers, respectively. An adaptation of Baker's theorem for linear forms in logarithms and Dujella and Pethő's reduction method confirms our main results.

Keywords: Fibonacci sequence; Jacobsthal sequence; linear forms in logarithms; reduction method.

1 Introduction

The Fibonacci sequence is an integer sequence A000045 defined recursively by $F_0 = 0$, $F_1 = 1$ and

$$F_{c+2} = F_{c+1} + F_c, \quad \text{where} \quad c \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}.$$

Debnath [7] outlines the historical background and applications of the sequence, while Koshy [19] presents modern identities and results that support current research. More generally, integer sequences, including the Fibonacci sequence, are cataloged in the On-Line Encyclopedia of Integer Sequences (OEIS) [21].

Similarly, the Jacobsthal sequence is an integer sequence A001045 defined by $J_0 = 0$, $J_1 = 1$ and

$$J_{c+2} = J_{c+1} + 2J_c, \quad \text{where} \quad c \in \mathbb{N}_0.$$

Horadam [17] introduced Jacobsthal representation numbers and later extended this approach to representation polynomials [18]. He also explored the relationship between Jacobsthal and Pell sequences in the context of curves [16]. Further the developments include generalized Jacobsthal polynomials by Djordjevic [8] and order-k Jacobsthal numbers studied by Yilmaz and Bozkurt [23]. Most recently, Aydın [4] presented broad generalizations of the Jacobsthal sequence within number theory.

Abdulzahra and Hashim [1] investigated solutions of a Diophantine equation in Lucas sequences, employing Baker's method and reduction arguments. Similarly, Alabrahimi and Hashim [2] studied the Jin–Schmidt equation within Lucas numbers, also using linear forms in logarithms to derive explicit solutions.

Intensive study is done on finding all terms of some integer sequences which are sums, products, and differences of two terms of an integer sequence. For example, all Fermat and Mersenne numbers that can be expressed as the product of two Jacobsthal numbers were discovered in [15]. Rihane [22] identified all k-Fibonacci numbers that are the product of two Balancing or Lucas-Balancing numbers. Emin [10] found all Pell (or Mersenne) numbers that are expressible as the summing of two arbitrary Mersenne (or Pell) numbers. Erduvany and Keskinz [12] found all Fibonacci numbers and Lucas numbers which can be expressed as the product of two Jacobsthal-Lucas numbers. Gaber and Ahmed [13] found all balancing numbers and Lucas balancing numbers that are expressible as the sum of two arbitrary Jacobsthal numbers. Alan and Alan [3] found all Mersenne numbers that can be expressed as the sum of two Pell numbers. All Fibonacci numbers that can be expressed as the sums of two Padovan numbers were discovered in [14]. In 2021, Erduvan and Keskin [11] solved the two Diophantine equations,

$$F_c = J_a J_b, \quad \text{and} \quad J_c = F_a F_b,$$

in non-negative integers a, b and c.

As an extension of those works, we study the two Diophantine equations,

$$J_a + J_b = F_c, \tag{1}$$

and

$$F_a + F_b = J_c, \tag{2}$$

in non-negative integers a, b and c. Our results are the following:

Theorem 1.1. *The set of non-negative integer solutions* (a, b, c)*, with* $a \geq b$*, to the Diophantine equation* (1) *is the following,*

$$\left\{ \begin{array}{l} (0,0,0),\ (1,0,1),\ (1,0,2),\ (1,1,3),\ (2,0,1),\ (2,0,2),\ (2,1,3), \\ (2,2,3),\ (3,0,4),\ (4,0,5),\ (4,3,6),\ (6,0,8) \end{array} \right\}.$$

Theorem 1.2. *The set of non-negative integer solutions* (a, b, c)*, with* $a \geq b$*, to the Diophantine equation* (2) *is the following,*

$$\left\{ \begin{array}{l} (0,0,0),\ (1,0,1),\ (1,0,2),\ (2,0,1),\ (2,0,2),\ (3,1,3),\ (3,2,3), \\ (4,0,3),\ (4,3,4),\ (5,0,4),\ (6,4,5),\ (7,6,6),\ (8,0,6) \end{array} \right\}.$$

Our method involves the use of one of Matveev's most well-known variations [20] of Baker's theory to constrain all implicit variables to one main variable. Matveev's theorem will help us in obtaining maximum values for c and a, however these values are very high. Then, in order to lower these upper bounds, we employ the reduction lemma established by Dujella and Pethő [9]. Finally, we perform computations using computer algebra systems (CAS) such as Sage and Mathematica to determine and verify all solutions.

This work is organized as follows. We go over a few facts that are essential to our proofs in Section 2. The proof of Theorem 1.1 is presented in Section 3. The proof of Theorem 1.2 is shown in Section 4. The conclusion is in Section 5.

2 Preliminary Results

2.1 Fibonacci sequence

The Binet formula for the Fibonacci sequence is

$$F_c = \frac{\alpha^c - \beta^c}{\sqrt{5}}, \qquad c \geq 0, \tag{3}$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \qquad \beta = \frac{1-\sqrt{5}}{2},$$

are the roots of the auxiliary equation $x^2 - x - 1 = 0$. The following inequality will help us find relations between the three variables a, b and c,

$$\alpha^{c-2} \leq F_c \leq \alpha^{c-1}, \qquad c \geq 1. \tag{4}$$

2.2 Jacobsthal sequence

Jacobsthal sequence has the characteristic equation,

$$x^2 - x - 2 = 0,$$

which has the roots 2, and -1, and its Binet formula is given by,

$$J_c = \frac{2^c - (-1)^c}{3}, \qquad c \geq 0. \tag{5}$$

The c^{th} Jacobsthal number satisfies the following inequalities,

$$2^{c-2} \leq J_c \leq 2^{c-1}, \qquad c \geq 1, \tag{6}$$
$$J_c < 2^{c-1}, \qquad c \geq 2. \tag{7}$$

2.3 Logarithmic linear forms

Given an algebraic number λ of degree r, let λ have the minimal polynomial as follows,

$$c_0 x^r + c_1 x^{r-1} + \cdots + c_d = c_0 \prod_{i=1}^{r} \left(x - \lambda^{(i)}\right) \in \mathbb{Z}[x],$$

where $\lambda^{(i)}$'s are the conjugates of λ, $c_0 > 0$, and c_i's are relativity prime integers. The logarithmic height h of λ is defined by,

$$h(\lambda) = \frac{1}{r}\left(\log c_0 + \sum_{i=1}^{r} \log\left(\max\left\{\left|\lambda^{(i)}\right|, 1\right\}\right)\right).$$

The function h possesses the following important properties;

$$h(\lambda_1 \pm \lambda_2) \leq h(\lambda_1) + h(\lambda_2) + \log 2,$$
$$h\left(\lambda_1 \lambda_2^{\pm 1}\right) \leq h(\lambda_1) + h(\lambda_2),$$
$$h(\lambda^s) = |s| \; h(\lambda) \quad (s \in \mathbb{Z}).$$

These properties will be applied in the upcoming sections without particular references.

Based on Matveev's Corollary 2.3 [20], Bugeaud, Mignotte, and Siksek [6] derived the subsequent theorem.

Theorem 2.1. *Let F be a real algebraic number field of degree T, $\lambda_1, \ldots, \lambda_r$ be positive real algebraic numbers in F, and $v_1, \ldots, v_r$ be non zero integers where,*

$$\Lambda = \lambda_1^{v_1} \cdots \lambda_t^{v_r} - 1 \neq 0.$$

Then,

$$\log|\Lambda| > -1.4 \times 30^{r+3} \times r^{4.5} \times T^2 \,(1 + \log T)(1 + \log V)\, G_1\, G_2 \ldots G_r, \tag{8}$$

where

$$V \geq \max\{|v_1|, \ldots, |v_r|\},$$

and $G_i \geq \max\{T\, h(\lambda_i), |\log \lambda_i|, 0.16\}$ for all $i = 1, \ldots, r$.

The last theorem will be utilized to provide upper bounds for our variables.

The subsequent lemma, which is a modification of a lemma by Baker and Davenport [5], was demonstrated by Dujella and Pethő [9]. The distance to the closest integer is referred by $\|\cdot\|$.

Lemma 2.1. *Assume that K is a positive integer. For the irrational number λ, assume that p/q is a convergent of the continued fraction of λ such that $q > 6K$. Assume that $S > 0$, $H > 1$, and μ are some real numbers. Let $\epsilon = \|\mu\, q\| - K\,\|\lambda\, q\|$. If $\epsilon > 0$, and (x, y, z) is a positive integer solution to the inequality,*

$$0 < |x\,\lambda - y + \mu| < S\,H^{-z},$$

with $x \leq K$, then,

$$z < \frac{\log(Sq/\epsilon)}{\log H}. \tag{9}$$

The last lemma will be used to lower the upper bound that Theorem 2.1 yieldes.

3 Proof of Theorem 1.1

Proof. Let $a \geq b$. Using CAS, one can find that the only solutions to (1) within $0 \leq b \leq a \leq 200$ are those solutions that are mentioned in Theorem 1.1. So, it is possible to assume that $a > 200$. The proof is divided into three steps:

1) **Finding relations between c and a.**
For $b = 0$, one gets $J_a = F_c$. The last equality was studied in [11], and there are no solutions for $a > 200$. Since $J_1 = J_2$, it is possible to take $b \geq 2$.
Since $J_{i+2} = J_{i+1} + 2J_i$ where $i \geq 0$, $F_{i+2} = F_{i+1} + F_i$ where $i \geq 0$ and both sequences start with $0, 1, 1$, it is obvious that,

$$F_i < J_i, \qquad \text{for all } \ i \geq 3. \tag{10}$$

From (1), one obtains,

$$J_a < J_a + J_b = F_c. \tag{11}$$

Equations (10) and (11) show that,

$$a < c. \tag{12}$$

Hence, it is possible to take $c > 201$, $a > 200$ and $b \geq 2$.
Using the inequalities (4) and (7) with (1), one obtains,

$$\alpha^{c-2} \leq F_c = J_a + J_b \leq 2J_a < 2 \times 2^{a-1} = 2^a.$$

By taking the logarithm, one gets

$$c < \frac{\log 2}{\log \alpha} a + 2 < \frac{3}{2} a.$$

Therefore,

$$a < c < \frac{3}{2} a. \tag{13}$$

2) **Finding upper bounds of c and a.**
Equation (1) will now be reformulated as,

$$\frac{2^a}{3} - \frac{\alpha^c}{\sqrt{5}} = -J_b - \frac{\beta^c}{\sqrt{5}} + \frac{(-1)^a}{3}.$$

Consequently,

$$\left|\frac{2^a}{3} - \frac{\alpha^c}{\sqrt{5}}\right| < 2^{b-1} + \frac{7}{20}, \tag{14}$$

where $\frac{|\beta|^c}{\sqrt{5}} < 0.01$ for all $c \geq 201$, and $J_b < 2^{b-1}$ for all $b \geq 2$. Dividing (14) by $\frac{2^a}{3}$, one gets

$$\left|1 - 2^{-a}\alpha^c \frac{3}{\sqrt{5}}\right| < \frac{1}{2^{a-b}}\left(\frac{3}{2} + \frac{21}{20 \cdot 2^b}\right) < \frac{1.8}{2^{a-b}}, \tag{15}$$

since $b \geq 2$. Consequently,

$$\log\left|1 - 2^{-a}\alpha^c \frac{3}{\sqrt{5}}\right| < \log 1.8 - (a-b)\log 2. \tag{16}$$

Let,

$$\Lambda_1 := 1 - 2^{-a}\alpha^c \frac{3}{\sqrt{5}}.$$

In order to apply Theorem 2.1, we first prove that $\Lambda_1 \neq 0$. If $\Lambda_1 = 0$, one gets that $3\alpha^c = \sqrt{5} \cdot 2^a$. By conjugating the last equation, one gets $3\beta^c = -\sqrt{5} \cdot 2^a$. Therefore, $\sqrt{5} \cdot 2^a = |3\beta^c| < 3$, which is a contradiction.

Let,

$$\lambda_1 = 2, \qquad \lambda_2 = \alpha, \qquad \lambda_3 = \frac{3}{\sqrt{5}}, \qquad v_1 = -a, \qquad v_2 = c, \qquad v_3 = 1 \quad \text{and} \quad r = 3.$$

Since $\max\{|-a|, |c|, |1|\} = c$, it is possible to assign V the value of c. The smallest field that contains λ_1, λ_2 and λ_3 is $\mathbb{Q}(\sqrt{5})$, therefore, $T = 2$. The logarithmic heights are

$$h(\lambda_1) = \log 2,$$
$$h(\lambda_2) = \frac{1}{2}\log \alpha,$$

and

$$h(\lambda_3) = \frac{1}{2}\left(\log 5 + 2\log \frac{3}{\sqrt{5}}\right) = \log 3.$$

Applying Theorem 2.1 with $G_1 = 1.4$, $G_2 = 0.5$ and $G_3 = 2.2$, one gets,

$$\log|\Lambda_1| > -C(1 + \log c) \times 1.4 \times 0.5 \times 2.2, \tag{17}$$

where $C = 1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2) < 9.7 \times 10^{11}$.

Using (16) and (17) and the fact $2\log c > 1 + \log c$ for $c > 201$, one gets

$$\log 1.8 - (a-b)\log 2 > -2.99 \times 10^{12} \times \log c,$$

thus,

$$(a-b)\log 2 < 3 \times 10^{12} \times \log c. \tag{18}$$

The equation (1) is now rewritten as,

$$\frac{2^a}{3}+\frac{2^b}{3}-\frac{\alpha^c}{\sqrt{5}}=\frac{-\beta^c}{\sqrt{5}}+\frac{(-1)^a}{3}+\frac{(-1)^b}{3}.$$

Consequently,

$$\left|\frac{2^a}{3}+\frac{2^b}{3}-\frac{\alpha^c}{\sqrt{5}}\right|<0.01+\frac{1}{3}+\frac{1}{3}<\frac{7}{10}.$$

By rearranging the above inequality, one gets

$$\left|\frac{2^a}{3}\left(1+2^{b-a}\right)-\frac{\alpha^c}{\sqrt{5}}\right|<\frac{7}{10}.$$

Therefore,

$$\left|1-\alpha^c 2^{-a}\frac{3}{\sqrt{5}}\left(1+2^{b-a}\right)^{-1}\right|<\frac{21}{10}\cdot\frac{1}{2^a}. \tag{19}$$

Thus,

$$\log\left|1-\alpha^c 2^{-a}\frac{3}{\sqrt{5}}\left(1+2^{b-a}\right)^{-1}\right|<\log\frac{21}{10}-a\log 2. \tag{20}$$

Let,

$$\Lambda_2:=1-\alpha^c 2^{-a}\frac{3}{\sqrt{5}}\left(1+2^{b-a}\right)^{-1}.$$

If $\Lambda_2=0$, one gets $3\alpha^c=\sqrt{5}\left(2^a+2^b\right)$.
By conjugating the last equation, one gets $-\sqrt{5}\left(2^a+2^b\right)=3\beta^c$.
Therefore, $\sqrt{5}\left(2^a+2^b\right)=3\left|\beta\right|^c<3$, which is impossible.

Let,

$$\lambda_1=2,\quad \lambda_2=\alpha,\quad \lambda_3=\frac{3}{\sqrt{5}}\left(1+2^{b-a}\right)^{-1},\quad v_1=-a,\quad v_2=c,\quad v_3=1,\quad \text{and}\quad r=3.$$

Since λ_1, λ_2, and λ_3 are in $\mathbb{Q}(\sqrt{5})$, it is possible to take $T=2$. Also, we take $V=c$. The logarithmic heights are

$$h(\lambda_1)=\log 2,$$
$$h(\lambda_2)=\frac{1}{2}\log\alpha,$$

and

$$\begin{aligned}h(\lambda_3)=h\left(\frac{3}{\sqrt{5}}\left(1+2^{b-a}\right)^{-1}\right)&\le h\left(\frac{3}{\sqrt{5}}\right)+h\left(2^{b-a}\right)+\log 2\\&=\log 3+\left|b-a\right|\ h(2)+\log 2\\&=\log 6+(a-b)\log 2.\end{aligned}$$

Next, notice that,

$$\left|\log\lambda_3\right|=\left|\log\frac{3}{\sqrt{5}}-\log(1+2^{b-a})\right|<\log\frac{3}{\sqrt{5}}+\log 2<1.$$

Therefore, one can apply Theorem 2.1 with $G_1 = 1.4$, $G_2 = 0.5$ and $G_3 = 4 + 2(a-b)\log 2$, to get that,

$$\log|\Lambda_2| > -C(1+\log c) \times 1.4 \times 0.5 \times (4 + 2(a-b)\log 2),$$

where $C = 1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1+\log 2) < 9.7 \times 10^{11}$.

Since $2\log c > 1 + \log c$ for $c \geq 202$, one gets

$$\log|\Lambda_2| > -1.36 \times 10^{12} \times (\log c) \times (4 + 2(a-b)\log 2). \tag{21}$$

Using (20) and 21, we get

$$\log\frac{21}{10} - a\log 2 > -1.36 \times 10^{12} \times (\log c) \times (4 + 2(a-b)\log 2), \tag{22}$$

consequently,

$$a\log 2 < 1.37 \times 10^{12} \times (\log c) \times (4 + 2(a-b)\log 2). \tag{23}$$

Using (13), then,

$$c\log 2 < 2.055 \times 10^{12} \times (\log c)\,(4 + 2(a-b)\log 2)\,. \tag{24}$$

Substituting inequality (18) into inequality (24), one gets

$$c\log 2 < 2.055 \times 10^{12} \times (\log c)\left(4 + 6 \times 10^{12}\log c\right). \tag{25}$$

Using CAS, one gets

$$c < 10^{29}. \tag{26}$$

Therefore,

$$a < c < 10^{29}. \tag{27}$$

3) **Reducing the bound on a.**
Lemma 2.1 will now be used to lower the upper bound in inequality (27). Let,

$$\zeta_1 = c\log\alpha - a\log 2 + \log\frac{3}{\sqrt{5}}. \tag{28}$$

Equation (15) can be written as,

$$\left|e^{\zeta_1} - 1\right| < \frac{1.8}{2^{a-b}}. \tag{29}$$

Since $\Lambda_1 \neq 0$, one gets that $\zeta_1 \neq 0$. If $\zeta_1 > 0$, one gets

$$0 < \zeta_1 < e^{\zeta_1} - 1 = \left|e^{\zeta_1} - 1\right| < \frac{1.8}{2^{a-b}}. \tag{30}$$

If $\zeta_1 < 0$, and $a - b \geq 1$, one gets

$$1 - e^{\zeta_1} = \left|e^{\zeta_1} - 1\right| < 0.9,$$

hence,

$$e^{|\zeta_1|} = e^{-\zeta_1} < 10.$$

Therefore,

$$0 < |\zeta_1| < e^{|\zeta_1|} - 1 = e^{|\zeta_1|}(1 - e^{\zeta_1}) = e^{|\zeta_1|}\left|e^{\zeta_1} - 1\right| < \frac{18}{2^{a-b}}. \tag{31}$$

In both cases ($\zeta_1 > 0$ and $\zeta_1 < 0$), inequality (31) is true. By substituting the formula (28) for ζ_1 into the inequality above and dividing the result by $\log 2$, one obtains

$$0 < \left| c\frac{\log\alpha}{\log 2} - a + \frac{\log\frac{3}{\sqrt{5}}}{\log 2} \right| < \frac{26}{2^{a-b}}. \tag{32}$$

If $\dfrac{\log\alpha}{\log 2}$ were rational, there would exist non-zero integers x, y such that $\alpha^x = 2^y$. However, this equality cannot hold because α^x is irrational for all integers $x \neq 0$. To see this, observe that if α^x were rational, its conjugate β^x $\left(\text{where } \beta = \dfrac{1-\sqrt{5}}{2}\right)$ would satisfy $\alpha^x = \beta^x$, since the conjugate of a rational number is itself. This would imply,

$$\left(\frac{\alpha}{\beta}\right)^x = 1, \qquad \text{where} \qquad \frac{\alpha}{\beta} = -\frac{3+\sqrt{5}}{2}.$$

But $\left|\dfrac{\alpha}{\beta}\right| > 1$. Thus, α^x is irrational for all integers $x \neq 0$.

Next, Lemma 2.1 is applied to inequality (32). Let,

$$x = c, \qquad \lambda = \frac{\log\alpha}{\log 2}, \qquad \mu = \frac{\log\dfrac{3}{\sqrt{5}}}{\log 2}, \qquad S = 26, \qquad H = 2 \qquad \text{and} \qquad z = a - b.$$

Using inequality (27), it is possible to take $K = 10^{29}$ as an upper bound on c. For

$$q_{68} = 729778205193420675925701180216 > 6K,$$

where the denominator of the 68^{th} convergent of the continued fraction of λ is q_{68}, It is found that,

$$\epsilon = \|\mu\, q_{68}\| - K\,\|\lambda\, q_{68}\| > 0.266840003784733394316844.$$

Utilizing Lemma 2.1, one obtains

$$1 \leq a - b < \frac{\log(Sq/\epsilon)}{\log H} < 106. \tag{33}$$

Now, the case $a - b = 0$ is included. Therefore, it follows that,

$$0 \leq a - b < 106.$$

Using inequality (24) and the previously established maximum for $a - b$, then,

$$a < c < 1.7 \times 10^{16}. \tag{34}$$

To enhance the last upper bound of a, inequality (19) will be analyzed. Let,

$$\zeta_2 = c\log\alpha - a\log 2 + \log\frac{3}{\sqrt{5}}\left(1 + 2^{b-a}\right)^{-1}. \tag{35}$$

Equation (19) can be written as,

$$\left|1 - e^{\zeta_2}\right| < \frac{2.1}{2^a}. \tag{36}$$

Note that $\zeta_2 \neq 0$, since $\Lambda_2 \neq 0$. If $\zeta_2 > 0$, one gets,

$$0 < \zeta_2 < e^{\zeta_2} - 1 = \left|e^{\zeta_2} - 1\right| < \frac{2.1}{2^a}, \tag{37}$$

and if $\zeta_2 < 0$, one gets,

$$1 - e^{\zeta_2} = \left|e^{\zeta_2} - 1\right| < \frac{2.1}{2^a} < \frac{1}{2}, \qquad \text{for all } \ a > 200, \tag{38}$$

the last inequality implies that $e^{|\zeta_2|} < 2$. Hence,

$$0 < |\zeta_2| < e^{|\zeta_2|} - 1 = e^{|\zeta_2|}\left(1 - e^{\zeta_2}\right) = e^{|\zeta_2|}\left|e^{\zeta_2} - 1\right| < \frac{4.2}{2^a}. \tag{39}$$

In both cases, ($\zeta_2 > 0$, or $\zeta_2 < 0$), inequality (39) is true. By substituting the formula (35) for ζ_2 into the inequality above and dividing the result by $\log 2$, one obtains,

$$0 < \left|c\frac{\log \alpha}{\log 2} - a + \frac{\log \frac{3}{\sqrt{5}}\left(1 + 2^{b-a}\right)^{-1}}{\log 2}\right| < \frac{6.1}{2^a}. \tag{40}$$

Now, Lemma 2.1 is applied to the preceding inequality. Let,

$$\lambda = \frac{\log \alpha}{\log 2}, \qquad \mu = \frac{\log \frac{3}{\sqrt{5}}\left(1 + 2^{-(a-b)}\right)^{-1}}{\log 2}, \qquad S = 6.1, \qquad H = 2, \qquad \text{and} \qquad z = a.$$

Using inequality (34), it is possible to take $K = 1.7 \times 10^{16}$ as an upper bound on c. For

$$q_{41} = 1792876983842770837 > 6K,$$

where the denominator of the 41^{th} convergent of the continued fraction of λ is q_{41}. It was found that for all integers $a - b \in [0, 105]$, the smallest ϵ occurs at $a - b = 49$ and satisfies

$$\epsilon = \|\mu\, q_{41}\| - K\, \|\lambda\, q_{41}\| > 0.009770767.$$

By applying Lemma 2.1, one obtains

$$a < \frac{\log(Sq/\epsilon)}{\log H} < 70, \tag{41}$$

this is a counter to our presumption that $a > 200$. Note that the smallest ϵ was used in (41) because it produces the largest upper bound on a. Therefore, there is no solution to (1) when $a > 200$.

$\square$

4 Proof of Theorem 1.2

Proof. Let $a \geq b$. The cases $b = 0$ and $b = a - 1$ are clearly included as special cases of Theorem 1.1. Using CAS, one can find that there are no solutions of equation (2) within the range $0 \leq b \leq a \leq 200$, except the mentioned solutions in Theorem 1.2. Hence, throughout the proof, it is assumed that $a > 200$, $a - b \neq 1$ and $b \geq 2$, since $F_1 = F_2$. Since $b \neq 0$, we get $J_c > F_{200}$. Since $F_{199} < J_{139} < F_{200}$, we get $c > 139$.

In alignment with the demonstration of Theorem 1.1, the proof is divided into three separate stages, finding relations between c and a, finding upper bound of a, and reducing the bound on a.

1) Finding relations between c and a.
It is straightforward to prove by induction that,

$$\alpha^i < 2^{i-2}, \qquad \text{holds for all } \ i \geq 7.$$

Combining the last relation with (2), inequality (4), and inequality (7), one gets,

$$J_c < 2F_a \leq 2\alpha^{a-1} < 2^{a-2} \leq J_a, \qquad \text{holds for all } \ a > 200,$$

which implies that,

$$a > c. \tag{42}$$

2) Finding upper bound of a.
Equation (2) is now reformulated as,

$$\frac{\alpha^a}{\sqrt{5}} - \frac{2^c}{3} = -F_b - \frac{(-1)^c}{3} + \frac{\beta^a}{\sqrt{5}}.$$

Therefore,

$$\left| \frac{\alpha^a}{\sqrt{5}} - \frac{2^c}{3} \right| \leq F_b + \frac{|\beta|^a}{\sqrt{5}} + \frac{1}{3} < \alpha^{b-1} + \frac{7}{20},$$

where $\dfrac{|\beta|^a}{\sqrt{5}} < 0.01$ holds for every $a > 200$ and $F_b < \alpha^{b-1}$ holds for all $b \geq 2$. Dividing the above relation by $\dfrac{\alpha^a}{\sqrt{5}}$, we get,

$$\left| 1 - 2^c \alpha^{-a} \frac{\sqrt{5}}{3} \right| < \frac{1}{\alpha^{a-b}} \left(\sqrt{5} \times \alpha^{-1} + \frac{7 \times \sqrt{5}}{20 \times \alpha^b} \right) < \frac{1.7}{\alpha^{a-b}}. \tag{43}$$

Consequently,

$$\log \left| 1 - 2^c \alpha^{-a} \frac{\sqrt{5}}{3} \right| < \log 1.7 - (a - b) \log \alpha. \tag{44}$$

Let,

$$\Lambda_3 := 1 - 2^c \alpha^{-a} \frac{\sqrt{5}}{3}.$$

If $\Lambda_3 = 0$, one gets that $3\alpha^a = \sqrt{5} \times 2^c$.
By conjugating the last equality, one gets $3\beta^a = -\sqrt{5} \times 2^c$.
Therefore, $\sqrt{5} \times 2^c = 3\,|\beta|^a < 3$, which is impossible.

Let,

$$\lambda_1 = 2, \qquad \lambda_2 = \alpha, \qquad \lambda_3 = \frac{\sqrt{5}}{3}, \qquad v_1 = c, \qquad v_2 = -a, \qquad v_3 = 1, \qquad \text{and} \qquad r = 3.$$

Since $a > c$, it is possible to assign H the value of a and $F = \mathbb{Q}(\sqrt{5})$, which has a degree $T = 2$. The logarithmic heights are

$$h(\lambda_1) = \log 2,$$
$$h(\lambda_2) = \frac{\log \alpha}{2},$$

and

$$h(\lambda_3) = \log 3.$$

Applying Theorem 2.1 with $G_1 = 1.4$, $G_2 = 0.5$ and $G_3 = 2.2$, one gets,

$$\log |\Lambda_3| > -C(1 + \log a) \times 1.4 \times 0.5 \times 2.2,$$

where $C = 1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1 + \log 2) < 9.7 \times 10^{11}$. Using (44) and the fact that $2 \log a > 1 + \log a$ for every $a \geq 3$ one gets that,

$$\log 1.7 - (a - b) \log \alpha > -2.9876 \times 10^{12} \times \log a,$$

consequently,

$$(a - b) \log \alpha < 3 \times 10^{12} \log a. \tag{45}$$

Equation (2) is reorganized to utilize Theorem 2.1 once more, resulting in

$$\frac{\alpha^a}{\sqrt{5}} + \frac{\alpha^b}{\sqrt{5}} - \frac{2^c}{3} = \frac{\beta^a}{5} + \frac{\beta^b}{5} - \frac{(-1)^c}{3}.$$

Therefore,

$$\left| \frac{\alpha^a}{\sqrt{5}} \left(1 + \alpha^{b-a}\right) - \frac{2^c}{3} \right| \leq \frac{|\beta|^a + |\beta|^b}{\sqrt{5}} + \frac{1}{3} < 0.51,$$

since $\dfrac{|\beta|^a + |\beta|^b}{\sqrt{5}} < 0.17$, for all $a \geq 201$ and $b \geq 2$. Dividing by $\dfrac{\alpha^a}{\sqrt{5}} \left(1 + \alpha^{b-a}\right)$, one gets,

$$\left| 1 - 2^c \alpha^{-a} \frac{\sqrt{5}}{3} \left(1 + \alpha^{b-a}\right)^{-1} \right| < \frac{1.2}{\alpha^a}, \tag{46}$$

consequently,

$$\log \left| 1 - 2^c \alpha^{-a} \frac{\sqrt{5}}{3} \left(1 + \alpha^{b-a}\right)^{-1} \right| < \log 1.2 - a \log \alpha. \tag{47}$$

Let,

$$\Lambda_4 := 1 - 2^c\alpha^{-a}\frac{\sqrt{5}}{3}\left(1+\alpha^{b-a}\right)^{-1}.$$

If $\Lambda_4 = 0$, one would get that $3\left(\alpha^a + \alpha^b\right) = \sqrt{5}\times 2^c$.
By conjugating the last equation, one gets that $3\left(\beta^a + \beta^b\right) = -\sqrt{5}\times 2^c$.
Therefore, $\sqrt{5}\times 2^c = 3\left|\beta^a + \beta^b\right| < 3$ which is impossible.

Let,

$$\lambda_1 = 2, \quad \lambda_2 = \alpha, \quad \lambda_3 = \frac{\sqrt{5}}{3}\left(1+\alpha^{b-a}\right)^{-1}, \quad v_1 = c, \quad v_2 = -a, \quad v_3 = 1 \quad \text{and} \quad r = 3.$$

We take $H = a$, $F = \mathbb{Q}(\sqrt{5})$ and $T = 2$. The logarithmic heights are

$$\begin{aligned} h(\lambda_1) &= \log 2,\\ h(\lambda_2) &= \frac{1}{2}\log\alpha,\\ h\left(\frac{\sqrt{5}}{3}\left(1+\alpha^{b-a}\right)^{-1}\right) &\le h\left(\frac{\sqrt{5}}{3}\right) + h\left(1+\alpha^{b-a}\right)\\ &\le \log 3 + h\left(\alpha^{b-a}\right) + \log 2\\ &= \log 6 + |b-a|\, h(\alpha)\\ &= \log 6 + (a-b)\frac{\log\alpha}{2}. \end{aligned}$$

Next, we prove that $|\log\lambda_3| < 1$.

$$\begin{aligned} |\log\lambda_3| &= \left|\log\frac{\sqrt{5}}{3} - \log\left(1+\alpha^{b-a}\right)\right|\\ &< \left|\log\frac{\sqrt{5}}{3}\right| + \left|\log\left(1+\alpha^{b-a}\right)\right|\\ &< \log\frac{\sqrt{5}}{3} + \log 2 < 0.3 + 0.7 = 1. \end{aligned}$$

Applying Theorem 1.2, with

$$G_1 = 1.4, \qquad G_2 = 0.5 \qquad \text{and} \qquad G_3 = 4 + (a-b)\log\alpha > \max\left\{2h(\lambda_3), |\log\lambda_3|, 0.16\right\},$$

one gets,

$$\log|\Lambda_4| > -C\,(1+\log a)\times 1.4\times 0.5\times((a-b)\log\alpha + 4), \tag{48}$$

with $C = 1.4\times 30^6\times 3^{4.5}\times 2^2\times(1+\log 2) < 9.7\times 10^{11}$. Using (47) and (48) and the fact that $(1+\log a) < 2\log a$ holds for all $a \ge 3$, one gets,

$$\log 1.2 - a\log\alpha > -1.36\times 10^{12}\times\log a\times((a-b)\log\alpha + 4),$$

which implies that,

$$a\log\alpha < 1.37\times 10^{12}\times\log a\times((a-b)\log\alpha + 4). \tag{49}$$

Using (45) and (49), one gets,

$$a \log \alpha < 1.37 \times 10^{12} \times \log a \times (3 \times 10^{12} \times \log a + 4).$$

Using CAS, one gets,

$$a < 3.7 \times 10^{28}. \tag{50}$$

3) **Reducing the bound on a.**
The next step is to reduce the obtained upper bound in inequality (50) using Lemma 2.1. Let,

$$\zeta_3 = c \log 2 - a \log \alpha + \log \frac{\sqrt{5}}{3}. \tag{51}$$

Inequality (43) implies that,

$$\left|1 - e^{\zeta_3}\right| < \frac{1.7}{\alpha^{a-b}}.$$

It should be noted that $\zeta_3 \neq 0$ since $\Lambda_3 \neq 0$. For $\zeta_3 > 0$, one gets,

$$0 < \zeta_3 < e^{\zeta_3} - 1 = \left|e^{\zeta_3} - 1\right| < \frac{1.7}{\alpha^{a-b}}. \tag{52}$$

If $\zeta_3 < 0$, we suppose that $a - b \geq 2$ and since $\frac{1.7}{\alpha^{a-b}} < 0.65$ holds for $a - b \geq 2$, we get,

$$1 - e^{\zeta_3} = \left|1 - e^{\zeta_3}\right| < 0.65.$$

Consequently,

$$e^{|\zeta_3|} < 2.86.$$

Therefore,

$$0 < |\zeta_3| < e^{|\zeta_3|} - 1 = e^{|\zeta_3|}(1 - e^{\zeta_3}) = e^{|\zeta_3|}\left|e^{\zeta_3} - 1\right| < \frac{4.9}{\alpha^{a-b}}. \tag{53}$$

In both cases ($\zeta_3 > 0$ and $\zeta_3 < 0$), inequality (53) is true. By substituting the formula (51) for ζ_3 into the inequality above and dividing the result by $\log \alpha$, one obtains,

$$0 < \left| c\frac{\log 2}{\log \alpha} - a + \frac{\log \frac{\sqrt{5}}{3}}{\log \alpha} \right| < \frac{10.2}{\alpha^{a-b}}. \tag{54}$$

Let,

$$\lambda = \frac{\log 2}{\log \alpha}, \qquad \mu = \frac{\log \frac{\sqrt{5}}{3}}{\log \alpha}, \qquad S = 10.2, \qquad H = \alpha \quad \text{and} \quad z = a - b.$$

It should be noted that $\frac{\log 2}{\log \alpha}$ is irrational. This follows immediately from the established irrationality of $\frac{\log \alpha}{\log 2}$.

Using inequality (50), it is possible to take $K = 3.7 \times 10^{28}$ as an upper bound of c. For

$$q_{67} = 506642617699397667695263997821 > 6K = 2.22 \times 10^{29},$$

where the denominator of the 67^{th} convergent of the continued fraction of λ is q_{67}, it is found that,

$$\epsilon = \|\mu\, q_{67}\| - K\, \|\lambda\, q_{67}\| > 0.269087312907047.$$

Utilizing Lemma 2.1, then,

$$a - b < \frac{\log(Sq/\epsilon)}{\log H} < 149.7. \tag{55}$$

Now, the cases $a - b = 0$ and $a - b = 1$ are included. Therefore, it follows that,

$$0 \le a - b \le 149. \tag{56}$$

Using (56) and (49), we get

$$a < 8 \times 10^{15}. \tag{57}$$

Now, consider

$$\left| 1 - 2^c \alpha^{-a} \frac{\sqrt{5}}{3} \left(1 + \alpha^{b-a}\right)^{-1} \right| < \frac{1.2}{\alpha^a}.$$

Let,

$$\zeta_4 = c \log 2 - a \log \alpha + \log \frac{\sqrt{5}}{3} \left(1 + \alpha^{b-a}\right)^{-1}, \tag{58}$$

then,

$$\left|1 - e^{\zeta_4}\right| < \frac{1.2}{\alpha^a}. \tag{59}$$

Note that $\zeta_4 \neq 0$ since $\Lambda_4 \neq 0$. If $\zeta_4 > 0$, one gets,

$$0 < \zeta_4 < e^{\zeta_4} - 1 = \left|e^{\zeta_4} - 1\right| < \frac{1.2}{\alpha^a}. \tag{60}$$

Since $a > 200$, we have $\dfrac{1.2}{\alpha^a} < \dfrac{1}{2}$, which implies that for $\zeta_4 < 0$, we have

$$1 - e^{\zeta_4} = \left|e^{\zeta_4} - 1\right| < \frac{1}{2},$$

therefore,

$$e^{|\zeta_4|} = e^{-\zeta_4} < 2.$$

Thus,

$$0 < |\zeta_4| < e^{|\zeta_4|} - 1 = e^{|\zeta_4|}\left(1 - e^{\zeta_4}\right) = e^{|\zeta_4|}\left|e^{\zeta_4} - 1\right| < \frac{2.4}{\alpha^a}. \tag{61}$$

In both cases $(\zeta_4 > 0, \zeta_4 < 0)$, inequality (61) is true. By substituting the formula (58) for ζ_4 into the inequality above and dividing the result by $\log \alpha$, one obtains,

$$0 < \left| c\frac{\log 2}{\log \alpha} - a + \frac{\log \frac{\sqrt{5}}{3}(1 + \alpha^{b-a})^{-1}}{\log \alpha} \right| < \frac{5}{\alpha^a}. \tag{62}$$

Let,

$$\lambda = \frac{\log 2}{\log \alpha}, \qquad \mu = \frac{\log \frac{\sqrt{5}}{3}\left(1+\alpha^{b-a}\right)^{-1}}{\log \alpha}, \qquad S = 5, \qquad H = \alpha \qquad \text{and} \qquad z = a.$$

Using inequality (57), it is possible to take $K = 8 \times 10^{15} > a > c$ as an upper bound of c. For

$$q_{42} = 12446903481672945 63 > 6K = 4.8 \times 10^{16},$$

where the denominator of the 42^{th} convergent of the continued fraction of λ is q_{42}, it was found that for all integers $a - b \in [0, 149]$, the smallest ϵ occurs at $a - b = 29$ and satisfies,

$$\epsilon = \|\mu\, q_{42}\| - K\,\|\lambda\, q_{42}\| > 0.0112421970826642,$$

applying Lemma 2.1, one gets,

$$a < \frac{\log(Sq/\epsilon)}{\log H} < 99.3, \tag{63}$$

this is a counter to our presumption that $a > 200$. Note that the smallest ϵ was used in (63) because it produces the largest upper bound on a. Therefore, there is no solution to (2) when $a > 200$.

□

Remark 4.1. *When $b = 0$, the equations reduce to testing whether a term from one sequence appears in the other, such as $J_a = F_c$ or $F_a = J_c$. These special cases reveal the rare numbers that are both Fibonacci and Jacobsthal numbers, namely $0, 1, 3, 5,$ and 21.*

5 Conclusion

We have completely solved the Diophantine equations $J_a + J_b = F_c$ and $F_a + F_b = J_c$ through an innovative combination of Baker's theory (via Matveev's theorem) for initial bounds and the Dujella and Pethő reduction method for computational tractability. Our main results reveal finitely many solutions for both equations, along with finitely many intersections between Fibonacci and Jacobsthal numbers. For Theorem 1.1 ($J_a + J_b = F_c$), the largest solution is $J_6 + J_0 = F_8 = 21$. For Theorem 1.2 ($F_a + F_b = J_c$), multiple maximal solutions exist: $F_8 + F_0 = F_7 + F_6 = J_6 = 21$.

The special case $b = 0$ identifies all numbers that are simultaneously Fibonacci and Jacobsthal numbers: $0, 1, 3, 5,$ and 21. While most solutions occur at small indices due to the sequences' divergent growth rates ($J_n \sim 2^n$ vs. $F_n \sim \alpha^n$), these intersections represent rare arithmetic phenomena.

This work provides both complete solutions to these problems and a template for studying similar equations between recurrence sequences. The methods developed here could be extended to other sequence combinations in future research.

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Conflicts of Interest The authors declare no conflict of interest.

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